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NAVAL POSTGRADUATE SCHOOL

Monterey, California

A BI-MODAL INVENTORY STUDY
WITH RANDOM LEAD TIMES

by

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ABSTRACT:

A single-product continuous review inventory problem is formulated and solved. The chief virtue of the formulation is that the probability distribution of lead times is general. It is found that the optimal order size differs from the Wilson EOQ when holding costs or deterioration rate (both are lumped into a single discount rate) are large, and that it may even be a non-unique quantity. The second (fast) shipment mode enters in the same manner as a stockout cost.

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1. Background and Assumptions:

Sending cargo by air was a very expensive option until the end of World War II, at which time the sudden availability of pilots and surplus transport aircraft gave rise to costs that were low enough to make air freight a reasonable alternative for certain products. Since that time, the cost of air freight has fallen substantially relative to the cost of surface freight. For example, the ratio "cost per ton mile by air"/"cost per ton mile by rail" has fallen from about 20 in 1946 to about 6 in 1970. Introduction of the jumbo jets promises further reductions; in fact, air freight is already cheaper than truck freight on certain routes at certain times of day for certain products. The result of this is that air freight has been growing at about 15% per year for some time [5], which is explosive growth compared to the economy or to surface transportation. Furthermore, air freight still accounts for less than 1% of all domestic ton miles, which means that it could very well continue to outgrow total freight for some time to come.

Given the growth of the air freight industry and the expectation of further reductions in relative costs, the interest of major shippers in inventory models that contrast air freight and surface freight is natural [2]. Most of these models are "uni-modal" in the sense that the goal is to discover which mode results in the smallest (distribution + inventory + packaging + freight) cost, with the understanding that the cheapest mode should be used more or less exclusively. In general, air freight will overcome its cost handicap for products that are sufficiently fragile or perishable or expensive.

The original goal of the research presented here was to investigate a "bi-modal" inventory system, although this fact would perhaps not be obvious to the reader who skipped this introduction. By a "bi-modal" inventory system, we mean a system where a fast/expensive and a slow/cheap mode are both used as a matter of course, with the general idea being that the fast (air) mode is used only when the slow (surface) one "breaks down", as it will do in statistically predictable fashion. In the parlance of the trade, we are talking about "air express", rather than "air freight".

The simplest way of modeling surface "breakdowns" is to treat the surface lead times as independent random variables. There are two objections to the "independence" part, one practical and one theoretical. The practical objection is that many things that cause surface lead time delays are pervasive in nature, and thus affect several lead times simultaneously. In particular, strikes and seasonal demand peaks cause lead time randomness without necessarily permitting the order crossings that are inevitable if lead times are independent. The theoretical objection is that the independence assumption isn't even particularly convenient, since the resulting order crossings make analysis difficult. Nonetheless, the alternatives (multivariate lead time distributions or queueing models for the orders, for example) seem to present practical and theoretical difficulties that are even worse, so the lead times will be assumed to be independent in what follows.

Several schemes for dealing with the order crossing problem in continuous review systems have appeared. One is to assume that lead times are exponentially distributed [3], in which case a Markovian

analysis is possible. Another is to permit only unit orders, in which case it can be shown that the lead time distribution does not affect the ordering policy [4]. The assumption to be made here is somewhat unconventional. We will assume that each unit ordered can satisfy only one particular unit of demand. In a manufacturing context, this amounts to assuming that parts are not interchangeable. In a sales context, the corresponding assumption is that each item has been "colored" to suit the needs of a particular customer. The effect of the assumption is to decouple the orders so that it becomes immaterial whether they cross or not. Our results will apply rigorously only to such systems. If applied to conventional systems with interchangeable parts, errors will result to the extent that orders are likely to cross. The cost derived here will then constitute an upper bound on actual inventory cost.

Air freight will be assumed to have 0 lead time (see [1] for an analysis with positive lead times) and 0 set-up cost, so that the only charge is π per unit ordered. These assumptions definitely favor air freight over surface freight, and should be borne in mind in any applications. The effect of the assumptions is to make the cost of air freight an effective "penalty cost" that is presumably smaller than the penalty that would have to be paid if the unit actually failed to arrive. It can be seen that the resulting model will be bi-modal only by interpretation. An equally good description would be "a model including penalty costs applicable to the problem of ordering non-interchangeable parts". In fact, the terminology used in the Analysis section will be consistent with the idea of a penalty cost, since that

is the more elementary interpretation. However, the penalty cost that we have in mind is not so large as to justify analytical simplifications based on its size.

Some other assumptions are worthy of note. Demand is assumed to be continuous at U units per unit time. All carrying costs are assumed to be lumped into the discount rate, so that there will be no cost term, for example, that is proportional to average inventory on hand. The surface lead time is assumed to be non-negative with a finite mean.

Goals: We want to find a means for determining the optimal surface order quantity (Q) and the optimal amount of time (t) that an order should be placed before the first unit in it is needed. In addition, we want to determine conditions on the parameters of the problem and the distribution function of surface lead times $F(\cdot)$ such that

1) It is cheaper to pay the penalty than to order the unit (this would correspond to pure air freight in the bi-modal interpretation). See Section 3.

2) The optimal coefficient of π in the formula for total discounted cost is a discontinuous function of π (the coefficient is air freight usage in the bi-modal interpretation). See Section 4.

2. Analysis:

Imagine that a certain product is to be manufactured at the constant rate U, with each unit having a serial number and requiring a similarly numbered part from a supplier. It is desired to construct

an ordering scheme for obtaining parts from the supplier that minimizes the present value of the total cost of supply, where said cost includes a charge of $A + CQ$ when an order of size $Q > 0$ is made, discounted to the time when the order is made, and a charge of π for every unit produced without the subject part, discounted to the time when the deficient unit is produced. The parts are not interchangeable; a part that arrives late cannot be used on a unit with a different serial number. We assume $U, C, \pi > 0$, and $A \geq 0$.

Production starts at time 0, although it is possible to place orders in negative time. The lead times for all orders are independent, non-negative random variables with a common C.D.F. $F(\cdot)$ that is possibly defective in the sense that we only require $\lim_{t\to\infty} F(t) \equiv F(\infty) \leq 1$.

Let

V = minimum expected total cost of supply discounted to time 0, given that every part is ordered.

$\exp(-\alpha\tau)$ = discount factor at time τ.

$-t$ = time when first order is placed.

q = quantity of first order, in time units.

$c(t,q)$ = expected cost of first order, including penalties.

We require $q > 0$ and $t \geq 0$. Since the total cost of supply when nothing is ever ordered is $\int_0^\infty \pi U \exp(-\alpha t)dt = \pi U/\alpha \equiv V_0$, we will also restrict our attention to those cases where $V \leq V_0$.

We have

$$c(t,q) = \exp(\alpha t)(A+CUq) + \int_t^\infty \left\{ \int_0^{\min(u-t,q)} \pi U \exp(-\alpha v)dv \right\} d\,F(u)$$
$$+ \pi U\{(1-\exp(-\alpha q))/\alpha\}(1-F(\infty)) \tag{1}$$

The first term of (1) is the ordering cost, the second term is the expected penalty for being out of stock for a length of time that is at most q, and the third term is the penalty for loss of the order multiplied by the probability of loss. It will be convenient to change the form of the second term (call it I) through integration by parts. Let $g(v) = \pi U \exp(-\alpha v)$ and $h(u) = \min(u-t,q)$. Then we have

$$I \equiv \int_t^\infty \left\{ \int_0^{h(u)} g(v)dv \right\} dF(u) = F(u)\int_0^{h(u)} g(v)dv \Big|_t^\infty - \int_t^\infty F(u)h'(u)g(h(u))du \tag{2}$$

Since $h(t) = 0$, $h(\infty) = q$, and $h'(u) = 0$ for $u > t + q$, this is

$$I = F(\infty)\int_0^q g(v)dv - \int_t^{t+q} F(u)g(u-t)du, \tag{3}$$

$$I = \frac{\pi U}{\alpha}\left[(1-\exp(-\alpha q))F(\infty) - \alpha\int_t^{t+q} F(u)\exp(-\alpha(u-t))du\right] \tag{4}$$

The total discounted cost V is the sum of $c(t,q)$ plus the discounted cost of ordering all parts except for the first q. The latter quantity is V occurring at time q, so that $V \exp(-\alpha q)$ is the present value, and V must satisfy

$$V = \inf_{t,q} \{c(t,q) + V \exp(-\alpha q)\} \tag{5}$$

If we make the substitutions $x = \alpha q$, $y = \alpha t$, $G(v) = F(v/\alpha)$, and substitute (1) and (4) in (5), we have

$$V = \inf_{x,y}\left\{\exp(y)(A + \frac{CU}{\alpha}x) + \frac{\pi U}{\alpha}(1-\exp(-x) - \int_y^{y+x} G(v)\exp(y-v)\,dv) + V\exp(-x)\right\} \tag{6}$$

Let $a = \alpha A/\pi U$, $\delta = 1 - V/V_0$, and $r = C/\pi$. If V is subtracted from both sides of (6), and if the result is multiplied by $\alpha/\pi U = 1/V_0$, then (6) becomes

$$0 = \inf_{x,y}\left\{\exp(y)(a+rx) + \delta(1-\exp(-x)) - \int_y^{y+x} G(v)\exp(y-v)\,dv\right\} \tag{7}$$

Multiplying both sides by (-1), factoring out $\exp(y)$, and using basic properties of the exponential, we arrive at

$$0 = \sup_{x,y} \exp(y)\left\{\int_y^{y+x} g(v,r,\delta)\,dv - a\right\}, \tag{8}$$

$$\text{where} \quad g(v,r,\delta) = (G(v)-\delta)\exp(-v)-r \tag{9}$$

We are looking for solutions of (8) with $0 \leq \delta < 1$, since $0 < V \leq V_0$. If $\delta \geq 0$, then $G(v) - \delta \leq 1$, and it follows that $g(v,r,\delta) < 0$ for $v > \log(1/r)$. Also, $g(v,r,\delta) < 0$ for $v < 0$, since $G(v) = 0$ for $v < 0$. We can conclude that $\sup_{x,y}$ can be replaced by $\max_{x,y}$. Since $\exp(y)$ is bounded between positive numbers for $0 \leq y \leq \log(1/r)$, the bracketed factor in (8) must actually be 0, and we have the final form of the basic equation

$$a = \max_{x,y}\left\{\int_y^{y+x} g(v,r,\delta)\,dv\right\} \tag{10}$$

Let $\bar{r} \equiv \max G(v)\exp(-v)$. Then $g(v,r,0) \leq 0$ for all v if $r \geq \bar{r}$, and hence $g(v,r,\delta) < 0$ for all v if $r > \bar{r}$ and $\delta \geq 0$, since $g(v,r,\delta)$ is strictly decreasing in δ and r. It follows

that there is no solution of (10) with $\delta \geq 0$ if $r > \bar{r}$, even if $a = 0$. The interpretation of this result is that the part should not be supplied at all when $c/\pi \equiv r > \bar{r}$, even if the cost of ordering is 0.

Let $a(r,\delta)$ be the right hand side of (10). As long as $r \leq \bar{r}$ and $0 \leq \delta < 1$, the function $g(v,r,\delta)$ will have a maximum at some finite v for which $G(v) \geq \delta$. This maximum value will be a continuous, strictly decreasing function of δ (see lemma in Section 7) that is positive for $\delta = 0$ and negative for δ sufficiently close to 1. There is, therefore, a unique number δ_0 such that $\max_v g(v,r,\delta_0) = 0$. Since $g(v,r,1-r) = (G(v)-1)\exp(-v)-r(1-\exp(-v)) < 0$ for all v, $\delta_0 < 1 - r$. Evidently, $a(r,\delta_0) = 0$. We can now once again apply the lemma, with $S_u = [0,\delta_0)$, to conclude that $a(r,\delta)$ is continuous and strictly decreasing in δ. It follows that the equation $a(r,\delta) = a$ will have a unique solution $\delta(a)$ as long as $0 = a(r,\delta_0) < a \leq a(r,0)$. This solution represents the normalized total cost of supply. The optimal policy (not necessarily unique) can be recovered from any (x,y) pair that is optimizing in (10) when $\delta = \delta(a)$. If $a > a(r,0)$, there is no non-negative solution for δ, which means that the part should not be supplied.

The equation $a(r,\delta) = 0$ does not have a unique solution for δ, nor is there a maximizing, positive x. This simply indicates that the optimal order size is 0 when there is no ordering cost; the only questions are how far ahead of need each part should be ordered, and what the resultant cost is. When the ordering cost is 0, the cost per unit ordered is $c\exp(y) + \pi(1-G(y))$. If this is minimized and

multiplied by U/α, the result is the total discounted cost of supply: $V = (\pi U/\alpha)\min_y(r\exp(y) + (1-G(y)))$. By proceeding in the manner used when $a > 0$, with $\delta = 1 - V/V_0$, this can be reduced to $0 = \max_y g(y,r,\delta)$, the only solution of which is $\delta = \delta_0$. In other words, δ_0 is the normalized cost when $a = 0$, and the optimal normalized lead time is any maximizing y.

Intuitively, the function $\delta(a)$ should be such that $\lim_{a\to 0} \delta(a) = \delta_0$. To prove this, we first note that $\delta(a)$ is decreasing on $[0,a(r,0)]$, with $\delta(a) < \delta_0$, so that $\lim_{a\to 0} \delta(a) \equiv \delta(0)$ exists. Furthermore, we must have $\max_v g(v,r,\delta(0)) \leq 0$, since otherwise we would have $a(r,\delta(a)) > a$ for "a" sufficiently small. But always $\max_v g(v,r,\delta(a)) > 0$, so we must have $\lim_{a\to 0} \max_v g(v,r,\delta(a)) = \max_v g(v,r,\delta(0)) \geq 0$. It follows that $\max_v g(v,r,\delta(0)) = 0$, and, therefore, that $\delta(0) = \delta_0$.

3. A Graphical Method:

We can now describe the following graphical method for determining Q and t.

1) Construct a plot of $G(v)\exp(-v)$, where $G(v) = F(v/\alpha)$. This is illustrated in Figure 1 on p. 11 for a case where the lead time is either .6 or 1.2 (50/50) and $\alpha = 1$. This will be referred to as the Main Curve.

2) Compute $r = C/\pi$ and $\bar{r} = \max_v G(v)\exp(-v)$. If $r \geq \bar{r}$, the part should not be supplied, and $V = V_0 \equiv \pi U/\alpha$.

3) If the maximizing v is much less than 1.0, let

$Q = \sqrt{2A/\alpha CU}$ = Wilson EOQ, and choose t to minimize total discounted cost (see Section 5). Otherwise, go on to 4.

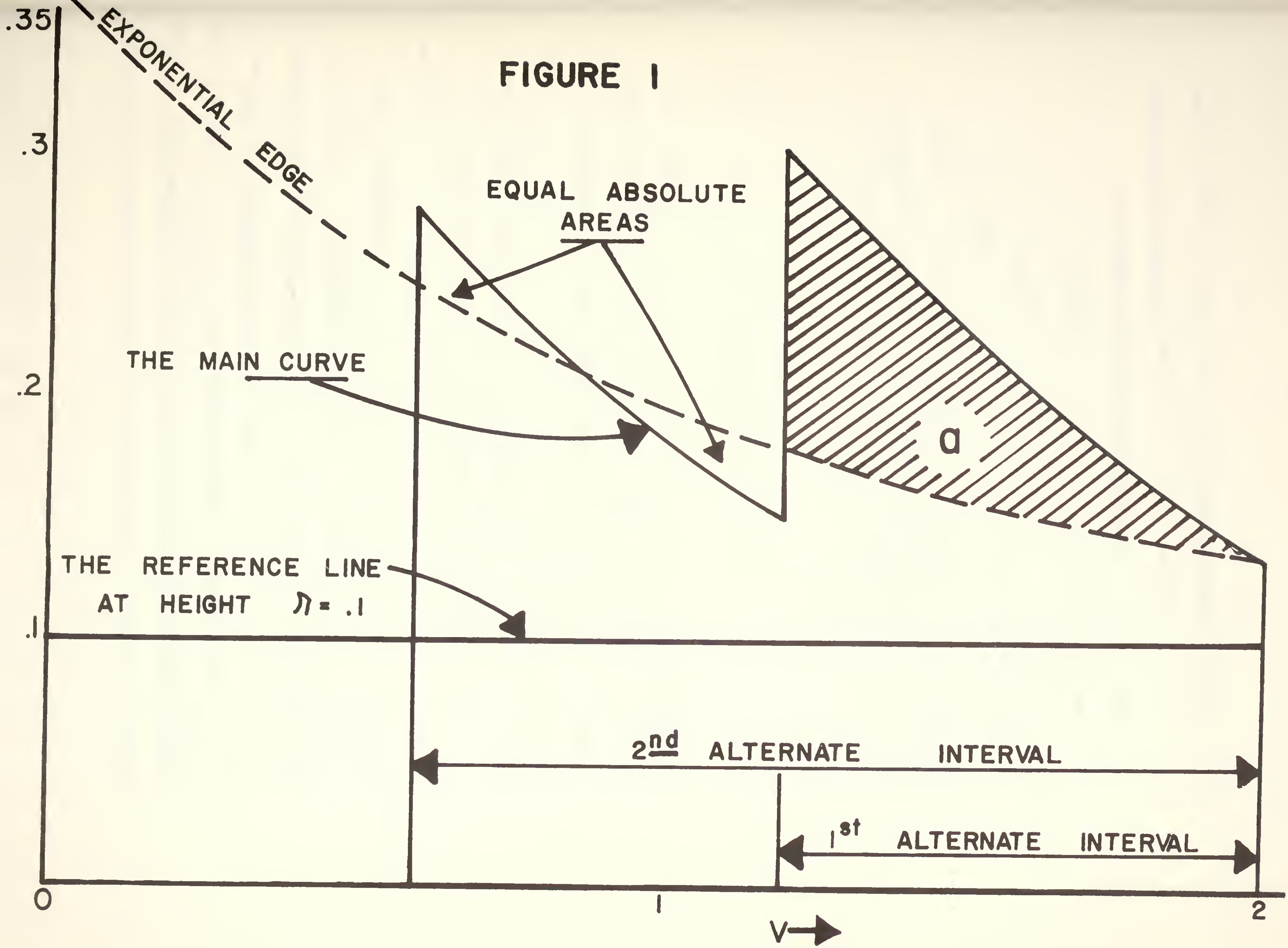
4) Construct a cutout of the exponential curve $\exp(-v)$. The vertical scale is technically immaterial, but subsequent computations will be aided if $\exp(-0)$ corresponds to about .5 on the $G(v)\exp(-v)$ curve.

5) Construct a horizontal "reference line" on the same plot as the Main Curve at height r (see Figure 1). Compute $a = \alpha A/\pi U$.

6) Slide the cutout along the reference line until the area above the difference between the Main Curve and the exponential edge of the cutout over some interval $[y, x+y]$ is "a". These two points y and $x + y$ will be intersections of the Main Curve with the exponential edge. They will normally be unique, although "a" has been chosen in Figure 1 to be the only value where they are not unique; either [.6, 1.99] or [1.2, 1.99] will work if "a" is the area shaded in Figure 1.

If "a" is so large that even sliding the cutout "all the way to the left" (this would make the exponential edge agree with the reference line) does not produce a large enough area, then the part should not be supplied.

FIGURE 1



EXPONENTIAL EDGE
EQUAL ABSOLUTE AREAS
THE MAIN CURVE
THE REFERENCE LINE AT HEIGHT $\eta = .1$
a
2^{nd} ALTERNATE INTERVAL
1^{st} ALTERNATE INTERVAL
.35
.3
.2
.1
0
1
2
V→

7) The optimal variables are $t = y/\alpha$ and $Q = xU/\alpha$. If V is desired, it can be obtained from the facts that $r + \delta$ is the intersection of the exponential edge with the vertical axis, and $V = V_0(1-\delta)$.

4. Nice Distributions and the Question of Continuity:

We now return to a question posed earlier: Under what circumstances can we expect the optimal coefficient of π in the expression for total discounted cost to be a continuous function of π? The coefficient of π after (6) is solved for V is $(U/\alpha)f(x,y)$, where

$$f(x,y) = 1 - \frac{\int_y^{y+x} G(v)e^{-v}dv}{\int_y^{y+x} e^{-v}dv} \tag{11}$$

Note that $f(x,y) = 0$ or 1 if $G(v)$ is 1 or 0 throughout the interval of integration, which corresponds to no penalty if the order is bound to arrive before need, and maximal penalty if the order will certainly not arrive in the interval of need. In general, $f(x,y)$ is the fraction of the maximum penalty that is paid, properly discounted.

With the other parameters fixed, x and y are functions of π, and our question is "Can we expect $f(x(\pi),y(\pi))$ to be a continuous function of π?". In general, the answer is "no". An example is shown in Figure 1, which is drawn for a particular value of π (call it π_0, whatever it is) for which (x,y) is not unique. If the first alternate interval is used, $f(x,y) = 0$, since $G(v) = 1$ throughout the interval. If the second alternate interval is used, then $f(x,y) > 0$ (the order has a 50/50 chance of arriving on time). If $\pi = \pi_0^+$, only the first alternate interval is optimal (the quantity of air freight decreases with its cost),

and if $\pi = \pi_0^-$, only the second alternate interval is optimal. This represents a discontinuity at π_0.

It should be evident that a discontinuity such as the one at π_0 could not have occurred if the Main Curve of Figure 1 had been so constructed that there could be no more than 2 intersections with the expcnential edge, no matter how the edge is oriented. Distribution functions $F(\cdot)$ with this property we will term "nice", since they correspond to situations where (x,y) is readily computed and continuous in the parameters. Analytically, the requirement is that the equation $g(v,r,\delta) \equiv (G(v)-\delta)\exp(-v) - r = 0$ should have no more than 2 roots v for any $r > 0$, $0 \leq \delta < 1$, and $\alpha > 0$, which is equivalent to requiring the same thing of the equation $f(t,r,\delta) \equiv (F(t)-\delta)\exp(-\alpha t) - r = 0$, where $t = v/\alpha$.

If $F(\cdot)$ is to be nice, it is evidently necessary that the function $F(t)\exp(-\alpha t)$ be unimodal (using the word in its mathematical sense) for all α, since otherwise more than two roots can be found for some α and $r > 0$ and $\delta = 0$. If we except cases where the lead time is actually deterministic, it follows that

<u>No discrete distribution of lead times is nice</u>.

On the other hand, suppose that $F(\cdot)$ actually has a density function $F'(t) = \frac{d}{dt} F(t)$, except possibly for a defect at $t = \infty$, and that there are at least four distinct roots of $f(t,r,\delta) = 0$ for some (r,δ,α). It follows that $h(t) \equiv f(t,r,\delta)\exp(\alpha t)$ also has at least four roots (the same ones), and that there are consequently at least three distinct roots of $h'(t) = 0$. Since $h'(t) = F'(t) - \alpha r \exp(\alpha t)$, this is the same as saying that $F'(t)\exp(-\alpha t) = \alpha r$ at three distinct

places. This is impossible if the function $F'(t)\exp(-\alpha t)$ is unimodal for all α, so that the assumption that $F'(t)\exp(-\alpha t)$ is unimodal for all α is sufficient to guarantee that there is no (r,δ,α) for which $f(t,r,\delta)$ has at least four roots. Furthermore, if there are exactly three roots for some (r,δ,α) then one of these roots must be a point of tangency, since $f(t,r,\delta)$ is negative for $t = 0$ and for t sufficiently large. It follows that there will be at least four roots either for $(r+\epsilon,\delta,\alpha)$ or $(r-\epsilon,\delta,\alpha)$, where ϵ is a small positive number. We have proved that three roots are essentially the same as four roots, and consequently that

<u>$F(\cdot)$ is nice if it is absolutely continuous except for a possible defect at $t = \infty$, and if $F'(t)\exp(-\alpha t)$ is unimodal for all α</u>.

It follows easily from this that

<u>All Gamma, Beta, Normal, and Uniform distributions of lead times are nice</u>.

Finally, consider a situation where $F(\cdot)$ is absolutely continuous and nice, but where "accidents" happen in a Poisson process with rate λ; if an accident happens while the shipment is in transit, then the shipment does not arrive. The new lead time is now defective, with the p.d.f. of the continuous part being $(F'(t))\exp(-\lambda t)$. This distribution will still be nice as long as $F(\cdot)$ is. In other words, the niceness property is robust in the face of Poisson-type accidents.

To conclude this discussion of continuity, we note that there will always be a discontinuity, even for nice lead time distributions, when π becomes so small that the penalty should be paid (only the fast mode

should be used) for all parts. The critical value of π satisfies the equation $a = a(r,0)$.

5. Robustness of the Wilson EOQ:

The Wilson EOQ has been shown to be optimal or near optimal in many circumstances other than those for which it was originally derived. The present problem provides another example of this phenomenon. The theorem below shows that the Wilson EOQ is optimal for small α, provided that the lead times have a finite mean when finite.

Let $1 - F(\infty)$ be the defect in the distribution of lead times. Since every part will be lost with probability $1 - F(\infty)$, one is naturally led to identify $C' \equiv C + \pi(1-F(\infty))$ as the cost per part; indeed, it is clear from the outset that the average rate of spending will approach $C'U$ in our model as α approaches 0, since the set-up cost will become negligible as the order size increases. Since the effective average rate of spending is also αV, we should expect to find $\lim_{\alpha\to 0} (\alpha V - C'U) = 0$ (the statement is somewhat stronger in the theorem below). But it is not clear whether q should be asymptotically $q_w \equiv \sqrt{2A/\alpha CU}$, or whether C should be replaced by C' in the EOQ formula. It will turn out that C should not be replaced by C'.

Theorem: Let $c(t,q)$ be as given by (1), let V be as given by (5), and let $q_w \equiv \sqrt{2A/\alpha CU}$. Assume $C' < \pi$, $\int_0^\infty [F(\infty) - F(t)]dt < \infty$ (this is the same as assuming that the mean lead time is finite except for the defect), and $A > 0$. Then $\lim_{\alpha\to 0} (\alpha V - C'U)/\sqrt{\alpha} = \lim_{\alpha\to 0} (\alpha V' - C'U)/\sqrt{\alpha} = \sqrt{2ACU}$, where V' is identical to V except that $q = q_w$.

Proof: Since $\lim_{\alpha\to 0} \bar{r} = F(\infty)$, and since $r < F(\infty)$ if and only $C' < \pi$, we will have $V = \min_{t,q} c(t,q)/(1-\exp(-\alpha q))$ and $V' = \min_{t} c(t,q_w)/(1-\exp(-\alpha q_w))$ when α is small enough; that is, the part will be supplied. From (1), since $\min(u-t,q) \le q$, we have

$$\exp(\alpha t)(A+CUq) + (\pi U/\alpha)(1-\exp(-\alpha q))(1-F(\infty)) \le c(t,q) \le \exp(\alpha t)(A+CUq) + (\pi U/\alpha)(1-\exp(-\alpha q))(1-F(t)) \tag{11}$$

From the second inequality in (11),

$$\alpha V' \le \alpha \exp(\alpha t) \frac{A+CUq_w}{1-\exp(-\alpha q_w)} + \pi U(1-F(t)) \tag{12}$$

Subtracting $C'U$ from both sides and dividing by $\sqrt{\alpha}$, we get $(\alpha V'-C'U)/\sqrt{\alpha} \le X + Y$, where

$$X = \exp(\alpha t) \frac{\sqrt{\alpha}(A+CUq_w)}{(1-\exp(-\alpha q_w))} - CU/\sqrt{\alpha}, \text{ and}$$

$$Y = \pi U(F(\infty)-F(t))/\sqrt{\alpha}$$

Let $t = \epsilon/\sqrt{\alpha}$, where $\epsilon > 0$. It follows from the finite mean assumption that $\lim_{\alpha\to 0} Y = 0$. Let $q_w = K/\sqrt{\alpha}$. Then (after applying L'Hospital's Rule) $\lim_{\alpha\to 0} X = A/K + CUK/2 + \epsilon CU$, which takes on its minimum value $\sqrt{2ACU} + \epsilon CU$ when $K = \sqrt{2A/CU}$. Since ϵ is arbitrary, we have shown that $\lim_{\alpha\to 0} (\alpha V'-C'U)/\sqrt{\alpha} \le \sqrt{2ACU}$, and that C should not be replaced by C' in the formula for the Wilson EOQ.

To complete the proof, we use the left hand inequality in (11), noting that $\exp(\alpha t) \ge 1$, to obtain

$$V \ge \min_{q\ge 0} \frac{A+CUq}{1-\exp(-\alpha q)} + (\pi U/\alpha)(1-F(\infty)) \equiv V_L$$

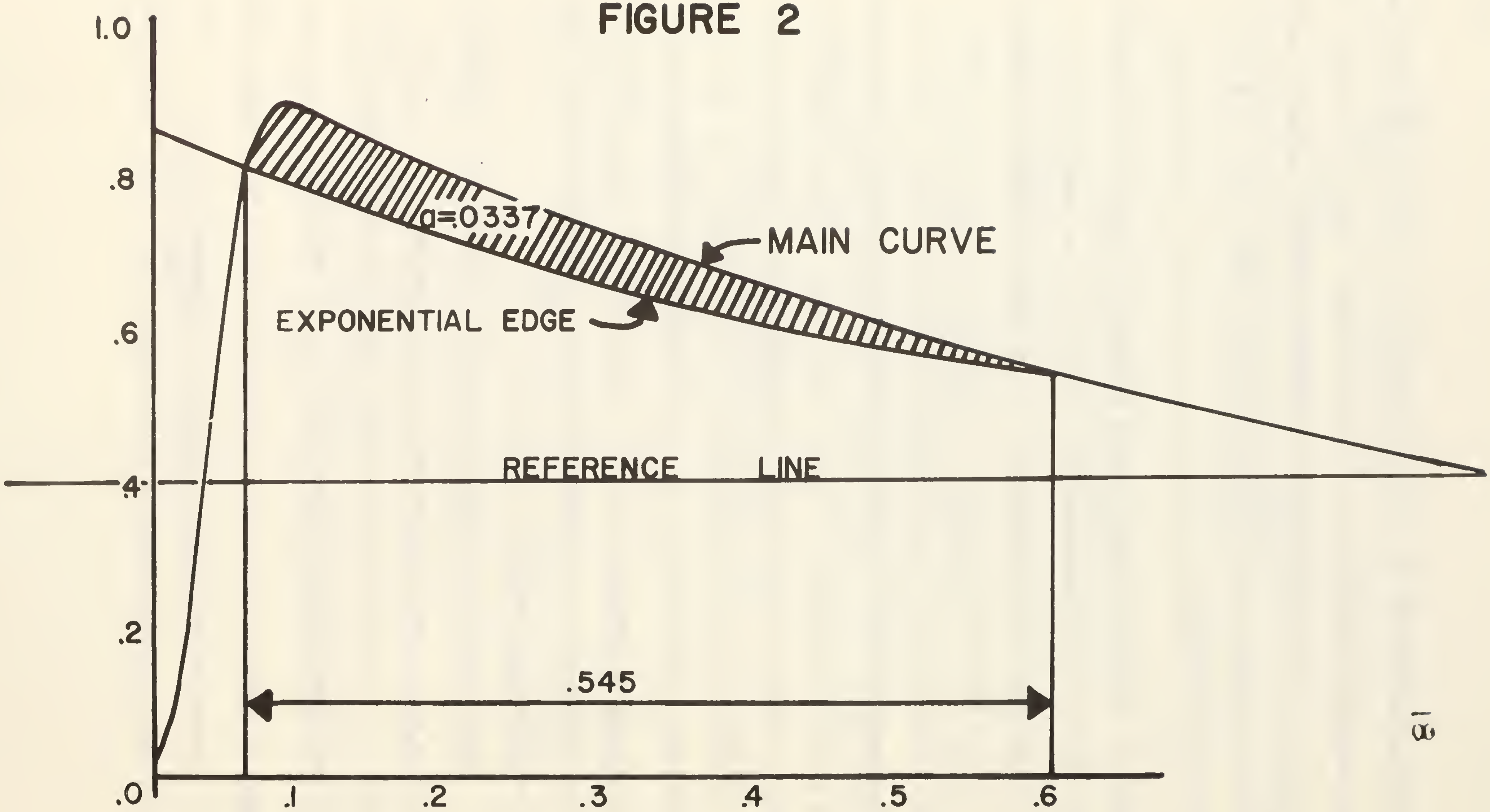
The derivative will be 0 at the minimizing q; solving $\frac{dV_L}{dq} = 0$, we obtain $(\exp(\alpha q)-1-\alpha q)/\alpha = A/CU$. It follows that $\lim_{\alpha\to 0} \alpha q^2 = 2A/CU$; i.e., $\lim_{\alpha\to 0} q/q_w = 1$. It is now simply a matter of repeating arguments used earlier to conclude that $\lim_{\alpha\to 0} (\alpha V_L - C'U)/\sqrt{\alpha} = \sqrt{2ACU}$. But $V_L \le V \le V'$; so the theorem follows.

Example 1: In order to test exactly how small α has to be before the Wilson EOQ is a good approximation, we will work an example where $\alpha = .2$ per year, the distribution of lead times is normal with mean .2 year and standard deviation .1 year, and $r = c/\pi = .4$. This leads to the Main Curve and Reference Line shown in Figure 2 on p. 18. The exponential edge in Figure 2 has been drawn arbitrarily, with "a" being specified implicitly. Using a planimeter, the area (dimensionless) between the Main Curve and the Exponential Edge is found to be $a = 0.337$, with the associated x and y being .545 and .063, respectively. The quantity αq_w is the Wilson EOQ in dimensionless form, and can be compared with x. In this problem, $\alpha q_w = \sqrt{2a/r} = .580$. This is less than 10% too large, and the percentage difference in total costs would be much smaller.

Given the fact that optima in inventory problems of this sort tend to be broad, and also the fact that the variable t is still available for optimization, there would seem to be little danger in using the Wilson EOQ in problems where $\alpha < .2$ per year.

Example 2: Suppose that $\alpha = 2$ and that the lead time is either .3 or .6 years, so that Figure 1 applies. The shaded area a is .444 in this case, and the resulting Wilson EOQ is $\alpha q_w = 2.9$. On the other

FIGURE 2



1.0
.8
.6
.4
.2
.0
.1
.2
.3
.4
.5
.6
a=.0337
MAIN CURVE
EXPONENTIAL EDGE
REFERENCE LINE
.545
$\bar{\alpha}$

hand, x can be either .8 (first alternate interval) or 1.4 (second alternate interval). The Wilson EOQ is much larger than either of the optimal order quantities.

6. Summary: The results that we have obtained are of only marginal practical importance, mainly on account of the fact that lead times in most practical problems tend to be much smaller than the reciprocal of the discount rate. Possible exceptions to this would be in problems where the discount rate is made artificially large to account for such things as product deterioration en route.

The principal contribution is conceptual: we have explored in depth a particular inventory problem that was simple enough to remove the need for making analytical approximations. We have discovered that the optimal order quantity is potentially non-unique--a fact that is of particular interest in the air freight interpretation. We have also found a class of lead time distributions, including the commonly used absolutely continuous distributions, for which this non-uniqueness is impossible. Finally, we have shown that the Wilson EOQ is a good approximation to the optimal order quantity when the discount rate is small.

7. Lemma: Let $f(u,v)$ be strictly decreasing and continuous in u for each v, let S_v be an arbitrary set, and let S_u be a possibly infinite interval of real numbers such that $g(u) \equiv \max_{v \in S_v} f(u,v)$ is well defined for $u \in S_u$. Then $g(u)$ is strictly decreasing and continuous on S_u.

Proof: Let $u_2 = u_1 + \delta$, where $\delta > 0$, and let v_1 and v_2 be maximizing for u_1 and u_2, respectively. Then $g(u_2) = f(u_2,v_2) < f(u_1,v_2) \leq f(u_1,v_1) = g(u_1)$, so $g(u)$ is strictly decreasing. Also, $g(u_2) \geq f(u_2,v_1) = g(u_1) + f(u_2,v_1) - f(u_1,v_1)$, so $\lim_{\delta \to 0} g(u_2) \geq g(u_1)$, so $g(u)$ is continuous.

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A single-product continuous review inventory problem is formulated and solved. The chief virtue of the formulation is that the probability distribution of lead times is general. It is found that the optimal order size differs from the Wilson EOQ when holding costs or deterioration rate (both are lumped into a single discount rate) are large, and that it may even be a non-unique quantity. The second (fast) shipment mode enters in the same manner as a stockout cost.

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